THE UNIVERSITY OF ALBERTA

ON NON-LINEAR ELECTROMAGNETIC THEORY WITH A BACK-GROUND

$$F_{ik,k} = -(1 + A_k^2)A_i - J_i$$

by

© MOHAN P. GOKHALE

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF PHYSICS

EDMONTON, ALBERTA

SPRING, 1971

ABSTRACT

The effect of a simple back-ground on the non-linear field theory proposed by Dr. H. Schiff is studied. To begin with a brief review of the work concerning original field equations, $F_{ik,k} = -(1 + A_k^2)A_i$, is given. The new equations, $F_{ik,k} = -(1 + A_k^2)A_i - J_i$, are proposed. As a simple back-ground, current J_i is chosen as $J_i = (0,0,0,i\rho)$ where ρ is a constant.

The static solutions with $\overline{H} = 0$ are studied. Neutral particle-like solutions are studied numerically and variationally. A possible situation for charged particle-like solutions is considered. The difficulty in the construction of a most general symmetric energy-momentum tensor with a local conservation law is pointed out. The mass spectrum of charged particles could not be established. The charge spectrum is established only heuristically. It is pointed out that a simple back-ground removes the mass degeneracy between ψ_+ and ψ_- neutral particle-like solutions. Possible consequences regarding the resulting inequality in positive and negative charge values are also investigated.

ACKNOWLEDGEMENT

I wish to express my gratitude to Dr. H. Schiff for supervising the work presented in this thesis. I am grateful also to the National Research Council of Canada for the financial support throughout the time of study and research.

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I. INTRODUCTION

The interest in non-linear field theories, as an approach towards the understanding of the intrinsic nature of the elementary particles is well known.[1]

Also there has been some interest in the hypothesis of a 'Universal Back-ground'. Recently, a number of physical phenomena, which hitherto could only be explained on the quantum hypothesis, have been explained entirely in terms of classical physics by introducing a zero-point radiation as a 'back-ground'.[2]

In the present work the idea of 'back-ground' is introduced in the classical non-linear theory proposed by Dr. H. Schiff.[3]

For the purpose of this work we take for 'back-ground' a constant uniform charge density throughout the space. The motivation behind this choice is to observe the effect of simple back-ground on the solutions of the non-linear system. This heuristic hypothesis may also be used as a basis for examining the ideas involving the inequality of basic positive and negative charges.[4]

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1. G. Mie, Ann. der Phy. 37, 511 (1912), 39, 1 (1912).
   M. Born and Infeld, Proc. Roy. Soc. London, A 144, 425 (1934)
   N. Rosen, Phy. Rev. 55, 94 (1939).
2. T.H. Boyer, Phy. Rev. D1 2257 (1970), 182, 1374 (1969) etc.
   T.W. Marshall, N.C. 38, 206 (1965).
3. H. Schiff, Proc. Roy. Soc. London, A 269, 277 (1962).
4. R.A. Lyttelton and H. Bondi, Proc. Roy. Soc. London, A 252, 313 (1959).
   R.W. Stover, T.I. Moran and J.W. Trischka, Phy. Rev. 164, 1599 (1967).

In the next chapter we take a brief look at the work concerning the original field equations,[5] while in the following chapters we study the new field equations.

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5. H. Schiff, Proc. Roy. Soc. London, A 269, 277 (1962).
G.W. Darewych, Ph. D. Thesis, U. of A. (1966).
G.W. Darewych and H. Schiff, Can. J. Phy., 47, 2597 (1969), etc.

II. SUMMARY OF THE PREVIOUS WORK

The original field equations are:

$$F_{ik,k} = -[(mc/\hbar)^2 + (g/c\hbar)^2 A_k^2] A_i \qquad \text{(II-1)}$$

where,

$$F_{ik} = A_{k,i} - A_{i,k}$$

$$A_i = (\vec{A}, i\phi)$$

$$x_i = (\vec{x}, ict)$$

and

$$A_{i,k} = \partial A_i / \partial x_k$$

$\overline{A}$ and ϕ are usual vector and scalar potentials respectively, m and g are the universal constants with dimensions of mass and charge respectively. Summation over repeated indices is assumed, with $i,j,k,\ldots = 1,2,3,4$.

Under the transformation $A_i \to A_i \cdot mc^2/g$ and $x_i \to x_i \cdot \hbar/mc$ the equations take the dimensionless form

$$F_{ik,k} = -(1 + A_k^2) A_i \qquad \text{(II-1a)}$$

In vector notation the field equations are

$$\bar{\nabla} \times \bar{H} = \partial \bar{E}/\partial t - (1 + \bar{A}^2 - \phi^2)\, \bar{A}$$

$$\bar{\nabla} \cdot \bar{E} = -(1 + \bar{A}^2 - \phi^2)\phi \qquad \text{(II-1b)}$$

The field equations (II-1) are equivalent to the principle of least action

$$\delta S = 0$$

with action function defined as

$$S = \int L(x_i, A_i, A_{i,k})\, d^4x$$

where the Lagrangian density is

$$L = -\frac{N}{4\pi}\left[\frac{1}{4} F_{ik}^2 + \frac{1}{2} A_i^2 + \frac{1}{4} A_i^2 A_k^2 + K\right] \qquad \text{(II-2)}$$

where N and K are contants.

The symmetrised energy - momentum tensor is

$$T_{ik} = -\frac{N}{4\pi}\left[L\delta_{ik} - F_{i\ell}F_{k\ell} - (1 + A_\ell^2)\, A_i A_k\right] \qquad \text{(II-3)}$$

The existance of non-Maxwellian plane wave-like solutions was indicated. But mainly the static solutions with $\overline{H} = 0$ were studied. Under this condition equations (II-1b) become

$$(1 + \overline{A}^2 - \phi^2)\,\overline{A} = 0 \qquad \text{(II-4)}$$

and

$$\nabla^2\phi = (1 + \overline{A}^2 - \phi^2)\phi \qquad \text{(II-5)}$$

This implies that $\overline{A} = 0$ or/and $(1 + \overline{A}^2 - \phi^2) = 0$. In the case, $\overline{A} = 0$ everywhere (II-5) becomes

$$\nabla^2\phi = \phi - \phi^3 \qquad \text{(II-6)}$$

It was shown that there exists a discrete set of particle-like solutions, which are asymptotic to zero as e^{-r}/r (for spherically symmetric case). These solutions represent spinless neutral particles (since $\int \nabla^2\phi d^3x = 0$) in this theory.

The equation was solved numerically and the masses of the particles were obtained by evaluating

$$\varepsilon = - \int T_{44}\, dV \qquad \text{(II-7)}$$

in the rest frame.

Also non-spherical solutions representing neutral particles were studied.[2]

In the case that $\bar{A} = 0$ in some spherical region V near the origin and $(1 + \bar{A}^2 - \phi^2) = 0$ outside V (both the conditions hold on the boundary surface S between these two regions), ϕ satisfies

$$\nabla^2\phi = \phi - \phi^3 \qquad \text{in} \qquad V \tag{II-8}$$

and

$$\nabla^2\phi = 0 \qquad \text{outside} \qquad V \tag{II-9}$$

with a boundary condition $\phi = \pm 1$ on S .

Spherically symmetric solutions of this system were found which represented charged particles. A spherically symmetric solution in the region outside V is[3]

$$\phi_{out} = a + \frac{b}{r} \ .$$

Constants a and b (the charge) are related by

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2. G.W. Darewych and H. Schiff, Can. J. Phy. 47, 1420 (1969).
3. The most general solution in this region may be described representing the electric multipole expansion of particle's charge distribution.

$$a + \frac{b}{R} = \pm 1$$

where R is the radius of V .

Now for each value of $\phi(0)$ there is a solution of (II-8) (constants a and b change with $\phi(0)$, the value of ϕ at $r=0$). These solutions can be grouped together according to the number of nodes they have on the lines $\phi = 0$ and $\phi = -1$, say M and N respectively. Thus there exists a continuum of well behaved solutions within each group of (M,N) which may be characterised by a parameter like $\phi(0)$ (or b). To determine a discrete set of particle-like solutions the principle of least action was invoked: only those solutions for which the total Lagrangian $L = \int \mathcal{L} dV$, ($\mathcal{L}$ being the Lagrangian density (II-2)), is a minimum with respect to the parameter characterising the continuum, will be considered to represent charged particles. Thus the first variation of L gives

$$\delta L = -b \ \delta a$$

so that we look for the solutions corresponding to the extrimum values of a (if they exist).

The existance of compound particles was also indicated and studied. If we divide the entire space in total n alternate linear $(A_i^2 = -1)$ and non-linear $(\vec{A} = 0)$ regions, the region near the origin being always non-linear, then we get neutral particles when n is odd and

charged particles when n is even.

Also the static interaction for widely separated particles was studied, showing that identical (non-identical, differing in sign only) neutral particles attract (repel) each other at large distances. For charged particles one gets the correct Coulomb Law.

Because of the tensor formulation the theory pertains only to Bosons.

The stability of neutral particle-like solutions (well behaved solutions of $\nabla^2\phi = f'(\phi)$) was studied. The solutions are unstable in the sense of Derrick's theorem.[5] Rosen[6] proved that, for a wide class of field equations, Derrick's instability criterion implies that the solutions are also dynamically unstable. This does not, however, immediately preclude the dynamical stability of the solutions of (II-6), since the Lagrangian density (II-2) does not belong to the class considered by Rosen. Still one might expect the neutral particle-like solutions to be unstable, since all known spinless elementary particles are only metastable. It was shown, though, that the neutral particle-like solutions are stable against many dissociation modes such as dissociation into plane waves and particles with $\overline{H} = 0$.

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5. G.H. Derrick, J.M.P. 5, 1252 (1965).

6. G. Rosen, J.M.P. 6, 1269 (1965).

Finally it may be pointed out that, in this theory, if ϕ is a solution so is $-\phi$. In case of charged particles they (ϕ and $-\phi$) represent oppositely charged particles with equal mass, while in case of neutral particles they represent non-identicle particles with same mass (or we can say that there is a mass degeneracy).

III. FIELD EQUATIONS

Introducing the back-ground J, we modify the equations (II-1) to the following form:

$$F_{ik,k} = -[(mc/\hbar)^2 + (g/\hbar c)^2 A_k^2] A_i - \hbar/g^2 J_i \qquad \text{(III-1)}$$

with the same notation as in chapter II.

Under the transformation

$$A_i \to A_i \cdot mc^2/g$$

$$x_i \to x_i \cdot \hbar/mc$$

$$J_i \to J_i \cdot cg/(\hbar/mc)^3$$

equations (III-1) take the dimensionless form

$$F_{ik,k} = -(1 + A_k^2) A_i - J_i \, . \qquad \text{(III-2)}$$

Now if we choose for back-ground a constant uniform charge density throughout the space i.e., $J = (0,0,0,i\rho)$, where ρ is a constant, then the corresponding vector equations become

$$\overline{\nabla} \times \overline{H} = \partial\overline{E}/\partial t - (1 + \overline{A}^{\,2} - \phi^2)\overline{A} \qquad \text{(III-3)}$$

$$\overline{\nabla} \cdot \overline{E} = -(1 + \overline{A}^2 - \phi^2)\phi - \rho \ . \qquad \text{(III-4)}$$

The conservation of the 4-current requires that

$$\frac{\partial}{\partial x_i} [(1 + A_j^2) A_i + J_i] = 0 \quad .$$

With $J = (0,0,0,i\rho)$, where ρ is time-independent $J_{i,1} = 0$ so that A_i must satisfy the gauge condition

$$\frac{\partial}{\partial x_i} [(1 + A_j^2) A_i] = 0 \ . \qquad \text{(III-5)}$$

Any solution of (III-2) will necessarily satisfy this condition.

The field equations (III-2) can also be considered equivalent to the principle of least action

$$\delta S = 0$$

with action function defined as

$$S = \int L(x_i, A_i, A_{i,k}, J_i) d^4x$$

where the Lagrangian density L is given by

$$L = \frac{-N}{4\pi}\left[\frac{1}{4} F_{ik}^2 + \frac{1}{2} A_i^2 + \frac{1}{4} A_i^2 A_k^2 + J_i A_i + K\right]$$

N and K being arbitrary constants.

The cannonical energy - momentum tensor defined as

$$T_{ik} = L\delta_{ik} - A_{j,i} \frac{\partial L}{\partial A_{j,k}} \qquad \text{(III-6)}$$

will be, using the Lagrangian density (III-5),

$$T_{ik} = \frac{-N}{4\pi} \{(\frac{1}{4} F_{\ell m}^2 + \frac{1}{2} A_\ell^2 + \frac{1}{4} A_\ell^2 A_m^2 + A_\ell J_\ell + K)\delta_{ik} - A_{j,i} F_{kj}\} \ . \qquad \text{(III-7)}$$

For the fields in the external current $J_i(x_k)$, the Lagrangian density, though invariant under the homogeneous Lorentz transformations, is not invariant under translations because of the explicit dependence of J_i on the co-ordinates x_i . As a result we do not have conserved energy - momentum for the total system. In fact, one can show that[1]

$$\begin{aligned} T_{ik,k} &= \frac{\partial L}{\partial x_i}\bigg|_{\text{explicit}} \\ &= A_k J_{k,i} \end{aligned}$$

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1. A.O. Barut, Electrodynamics and Classical Theory of Fields & Particles, p. 115 and 136.

where T_{ik} is the cannonical energy - momentum tensor.

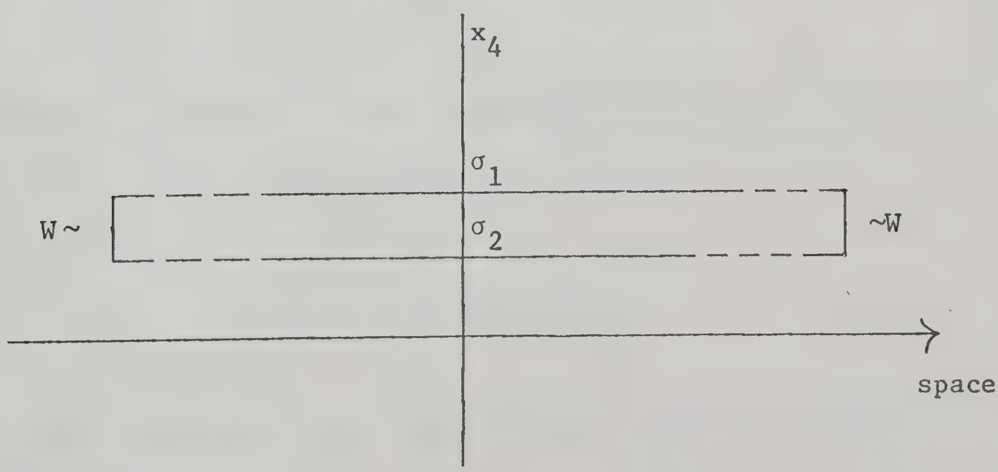
Now for the system we are considering, the external current density is simply $J = (0,0,0,i\rho)$ where ρ is a constant uniform charge density. So we do get local conservation of energy-momentum, that is to say

$$T_{ik,k} = 0 \quad . \tag{III-8}$$

But this need not immediately imply that we have a global conservation law. Because, if we condsider

$$0 = \int T_{ik,k}\, d^4x = \oint T_{ik}\, dS_k$$

and choose for the surface S_k, enclosing the 4-volume, two hyperplanes X_4 = constant, say σ_1 and σ_2, and the 'walls' (W) joining these two at spatial infinity, then



we have

$$\oint T_{ik}\, dS_k = 0 = \int_{\sigma_1} T_{i4}\, dS_4 - \int_{\sigma_2} T_{i4}\, dS_4 + \int_W T_{i\alpha}\, dS_\alpha \quad .$$

Thus we get a conservative law only if we could show that contribution from the last term is zero. In evaluating this term we must insert the values of the fields and the potentials at spatial infinity. Hence, in general, we can say that to have a conservation law the fields and the potentials must be such that $T_{i\alpha} \to 0$ fast enough as space co-ordinates $\to \infty$. Thus it is not sufficient that the fields alone vanish at infinity, but so should potentials, or at least be such that their contribution to the last term is zero.

For the static case, (III-8) reduces to

$$T_{i\beta,\beta} = 0$$

from which it follows that the solutions of (III-2) satisfy the integral relations

$$\int T_{i\gamma} dV = \oint x_\gamma T_{i\beta}\, df_\beta$$

where the integration on the right is over a surface enclosing the volume V . Thus the solutions for which $x_\gamma T_{i\beta} \to 0$ faster than $1/x_\gamma^2$, satisfy

$$\int T_{i\gamma}\, dV = 0 \tag{III-9}$$

where the indices $\alpha,\beta,\gamma\ \ldots = 1,2,3$.

IV. NEUTRAL PARTICLES

If we consider the static case with $\overline{H} = 0$ equations (III-3) and (III-4) become

$$(1 + \overline{A}^{\,2} - \phi^2)\,\overline{A} = 0 \tag{IV-1}$$

and

$$\nabla^2\phi = (1 + \overline{A}^{\,2} - \phi^2)\phi + \rho \ . \tag{IV-2}$$

Equation (IV-1) implies that $\overline{A} = 0$ or/and $(1 + \overline{A}^{\,2} - \phi^2) = 0$.

Now if we consider the particular case $\overline{A} = 0$ everywhere, then equation (IV-2) takes the form

$$\nabla^2\phi = \phi - \phi^3 + \rho \quad .$$

Consider the solution which can be expressed as

$$\phi = \psi + d$$

where the constant d is to be chosen suitably. We then have

$$\nabla^2\psi = (1 - 3d^2)\psi - 3d\psi^2 - \psi^3 + \rho + d - d^3 \quad .$$

A choice

$$d^3 - d = \rho$$

gives for ψ the following equation

$$\nabla^2\psi = (1 - 3d^2)\psi - 3d\psi^2 - \psi^3 \, . \tag{IV-3}$$

The phase-space analysis[1] along the lines of Finkelstein[2] et. al. suggests the existence of a discrete set of spherically symmetric solutions of (IV-3) with $\psi'(0) = 0$ and which are asymptotic to zero as $\frac{1}{r} \cdot \exp -(1 - 3d^2)^{1/2} r$. (This imposes a restriction on the values of d viz. $|d| < 1/(3)^{1/2}$) . And these solutions will then represent, in this theory, the neutral particles, since

$$\begin{aligned} \text{Total charge} &= -\frac{1}{4}\int \nabla^2\phi \, dV \\ &= -\frac{1}{4}\int \nabla^2\psi \, dV \\ &= 0 \, . \qquad (\because \ \psi \text{ decays faster than } 1/r) \end{aligned}$$

---

1. Phase-space analysis of (IV-3) is given in the Appendix A.
2. R. Finkelstein, R. LeLevier and M. Ruderman, Phy. Rev. 83, 326 (1951).

It may be pointed out that the condition $\psi'(0) = 0$ is necessary for the spherically symmetric solutions so that the electric field $\overline{E}$ is defined at the origin.

From (III-6) with $N = -1$,[3] we define the energy-momentum tensor as (with additional divergenceless term)

$$T_{ik} = L\delta_{ik} - A_{j,i} \frac{\partial L}{\partial A_{j,k}} + \frac{1}{4\pi} (A_i F_{k\ell})_{,\ell}$$

we get

$$T_{ik} = \frac{1}{4\pi} [(\tfrac{1}{4} F_{\ell m}^2 + \tfrac{1}{2} A_\ell^2 + \tfrac{1}{4} A_\ell^2 A_m^2 + J_\ell A_\ell + K)_{ik}$$

$$- F_{ij} F_{kj} - (1 + A_j^2) A_i A_k - A_i J_k] \quad . \qquad \text{(IV-4)}$$

The energy-momentum tensor thus obtained satisfies

$$T_{ik,k} = 0 \qquad \text{(IV-5)}$$

and is symmetric for the special case we are considering viz. $J = (0,0,0,i\rho)$ and $A = (0,0,0,i\phi)$.

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3. The choice $N = -1$ ensures positive mass for a neutral particle.

In fact for a static case with $\overline{A} = 0$ cannonical energy-momentum tensor itself is symmetric.[4] This is so because in this case theory is similar to a one component scalar theory.

Since $\overline{A} = 0$ and ψ goes as $e^{-\alpha r}/r$ asymptotically, as pointed out in chapter III, equation (III-9), components of $T_{i\gamma}$ (expressed in terms of ψ, and the proper choice of the constant K given below) satisfy

$$\int T_{i\gamma}\, d^3x = 0 \quad .$$

This implies that the energy-momentum tensor does have the correct particle-like transformations (viz. $\int T_{11}\, dV = \int T_{22}\, dV = \int T_{33}\, dV = 0$). [This is true of both, (IV-4) and the cannonical form (III-7) as well].[4]

Then the mass of the particle, evaluated in the rest frame, is

$$\varepsilon = -\int T_{44}\, dV = -\int T_{ii}\, dV \quad .$$

Thus for the finite mass of the particle the choice $4K = 3d^4 - d^2$ leaves us with the following expression for mass

$$M = \frac{1}{4\pi} \int [\psi^2 - (1 - 3d^2) d\, \psi]\, dV \; . \tag{IV-6}$$

---

4. See Appendix B, Sections I and II.

One interesting feature of the theory is that for a given eigenstate[5] (characterised by the number of nodes in ψ) we have two different solutions ψ_+ and ψ_- ($\neq -\psi_+$) depending upon whether $\psi(0)$ is positive or negative[6]. And these two solutions yield, in general, different masses M_+ and M_- , whereas for the case $\rho = 0$ we get $\psi_- = -\psi_+$ and $M_+ = M_-$. Thus the effect of a uniform background charge density is to remove the mass degeneracy between the neutral states ψ_+ and ψ_- .

Applying previously developed methods[7], solutions of (IV-3) were found to satisfy following integral relations:

$$\int \psi^3 \, dV = \int (1 - 3d^2)\psi \, dV - \int 3\, d\, \psi^2 \, dV \tag{IV-7}$$

$$\int (\nabla\psi)^2 \, dV = \int \psi^4 \, dV + \int 3\, d\, \psi^3 \, dV - \int (1 - 3d^2)\psi^2 \, dV \tag{IV-8}$$

$$\int (\partial\psi/\partial x)^2 \, dV = \int (\partial\psi/\partial x)^2 \, dV = \int (\partial\psi/\partial x)^2 \, dV = \int [F(\psi) - \tfrac{1}{2}\psi F'(\psi)] \, dV \tag{IV-9}$$

where

$$F'(\psi) = dF/d\psi \qquad \text{and}$$

$$F(\psi) = \frac{(1 - 3d^2)}{2}\psi^2 - d\psi^3 - \frac{1}{4}\psi^4 .$$

---

5. By eigenstate we refer to one of the solutions of (IV-3) forming a discrete set, asymptotic to zero.
6. This can be easily seen from the asymmetry in the phase diagram for $d \neq 0$ (Appendix A fig. A-2) as opposed to the symmetry of the one for $d = 0$ (Appendix A fig. A-1).
7. G.W. Darewych and H. Schiff, J.M.P. 8, 1479 (1966).

From (IV-9) follows

$$\int (\nabla\psi)^2 \, dV = \int \frac{3}{2} \psi^4 \, dV + \int 6 \, d \, \psi^3 \, dV - \int 3(1 - 3d^2)\psi^2 \, dV \, . \quad \text{(IV-10)}$$

Relations (IV-8) and (IV-10) can be combined to give

$$\int \psi^4 \, dV = \int 4(1 - 3d^2) \, \psi^2 \, dV - \int 6 \, d \, \psi^3 \, dV \quad . \qquad \text{(IV-11)}$$

Spherically symmetric solutions, for ground state (nodeless) and first excited state (one node), of (IV-3) were obtained numberically, for various values of d, using the method of central differences. It was verified numerically that these solutions satisfy the relations (IV-7) and (IV-11). The required integrals were evaluated by using Simpson's 1/3 rule. All the numerical results in this work were obtained on the IBM 360/67 computer at the Department of Computer Science, U. of A. To minimise the error due to rounding Double Precision was used throughout.

For solving the equation (IV-3) following transformation was used,

$$f = r \, \psi$$

so that the equation for f becomes

$$f'' = (1 - 3\delta^2)\, f - 3\, \delta\, f^2/r - f^3/r^2 \quad .$$

Further, the substitution

$$t = 1 - e^{-r}$$

was made, so that t is confined to the finite range from 0 to 1 as r goes from 0 to ∞, and $f = 0$ at $t = 0$ and at $t = 1$.

Numerical results thus obtained are tabulated on the next page, following which is a graphical plot nodeless ψ_+ and $-\psi_-$ for $\delta = 0$ and $\delta = 0.07$, and one node ψ_+ and $-\psi_-$ for $\delta = 0.06$. Mass (M) is in units of $mc\hbar/g^2$, ψ is in units of mc^2/g and r is in units of $\hbar/mc$.

A variational calculation was also carried out for spherically symmetric and non-spherical (viz. axially symmetric) states as well and is given in the next chapter.

d	M_+	M_-	M_+/M_-	Truncation error in $\int\psi^2$ (leading term in $M_{\pm}$)	Integral relation (IV-11)	Integral relation (IV-7)
0	1.503788		1.00	-4×10^{-10}	.00027%	.34%
		1.503788		-4×10^{-10}	.00027%	.34%
10^{-10}	1.503788		differs from 1 in 9th decimal place	-4×10^{-10}	.00027%	.34%
		1.503788		-3×10^{-10}	.00027%	.34%
10^{-8}	1.503787		differs from 1 in 7th decimal place	-5×10^{-10}	.00027%	.34%
		1.503788		-9×10^{-10}	.00027%	.34%
10^{-5}	1.50371		differs from 1 in 4th decimal place	-2×10^{-10}	.00027%	.34%
		1.503863		-1×10^{-11}	.00027%	.34%
10^{-3}	1.496244		differs from 1 in 3rd decimal place	-3×10^{-10}	.00027%	.34%
		1.511324		-2×10^{-10}	.00027%	.34%
7×10^{-2}	0.958131		0.476	-1×10^{-10}	.00021%	.37%
		2.011828		5×10^{-10}	.00036%	.35%

TABLE (IV-1)

d	M_+	M_-	M_+/M_-	Truncation error in $\int\psi^2$ (leading term in $M_\pm$)	Integral relation (IV-11)	Integral relation (IV - 7)
0	9.467971		1.0	-0.2×10^{-8}	0.0012%	0.50%
		9.467971		-0.2×10^{-8}	"	"
10^{-10}	9.467971		differs from 1 in 10th decimal place	-0.1×10^{-8}	"	"
		9.467971		-0.25×10^{-9}	"	"
10^{-8}	9.467971		differs from 1 in 8th decimal place	-0.26×10^{-9}	"	"
		9.467971		-0.13×10^{-8}	"	"
10^{-5}	9.468294		differs from 1 in 5th place	-0.98×10^{-11}	"	"
		9.467648		-0.68×10^{-10}	"	"
10^{-3}	9.500		differs from 1 in 3rd place	-0.6×10^{-11}	"	"
		9.435637		-0.2×10^{-8}	"	"
6×10^{-2}	11.3167		1.52	0.19×10^{-7}	0.0015%	0.5%
		7.445		0.3×10^{-7}	0.0008%	0.56%

TABLE (IV - 2)

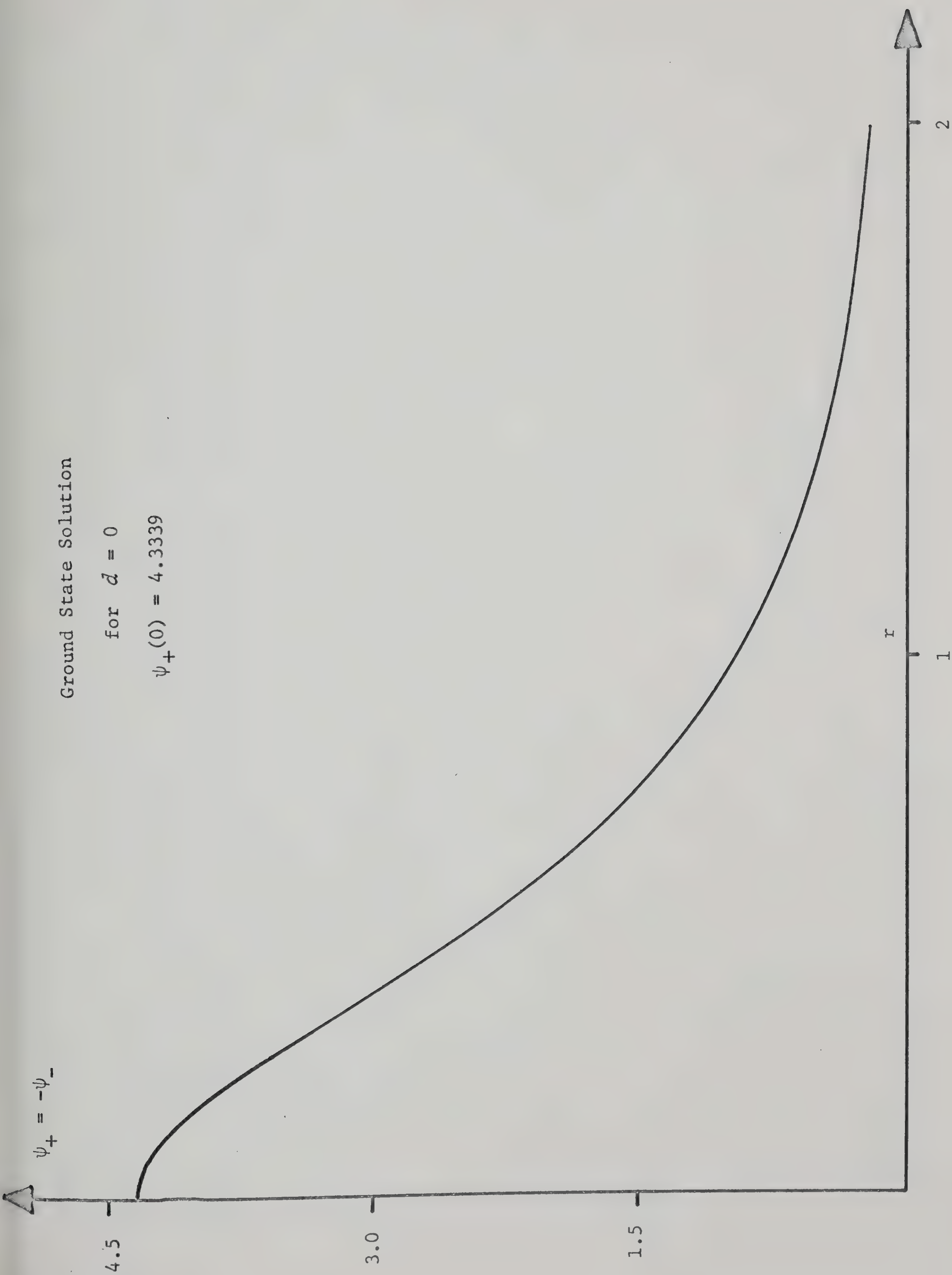
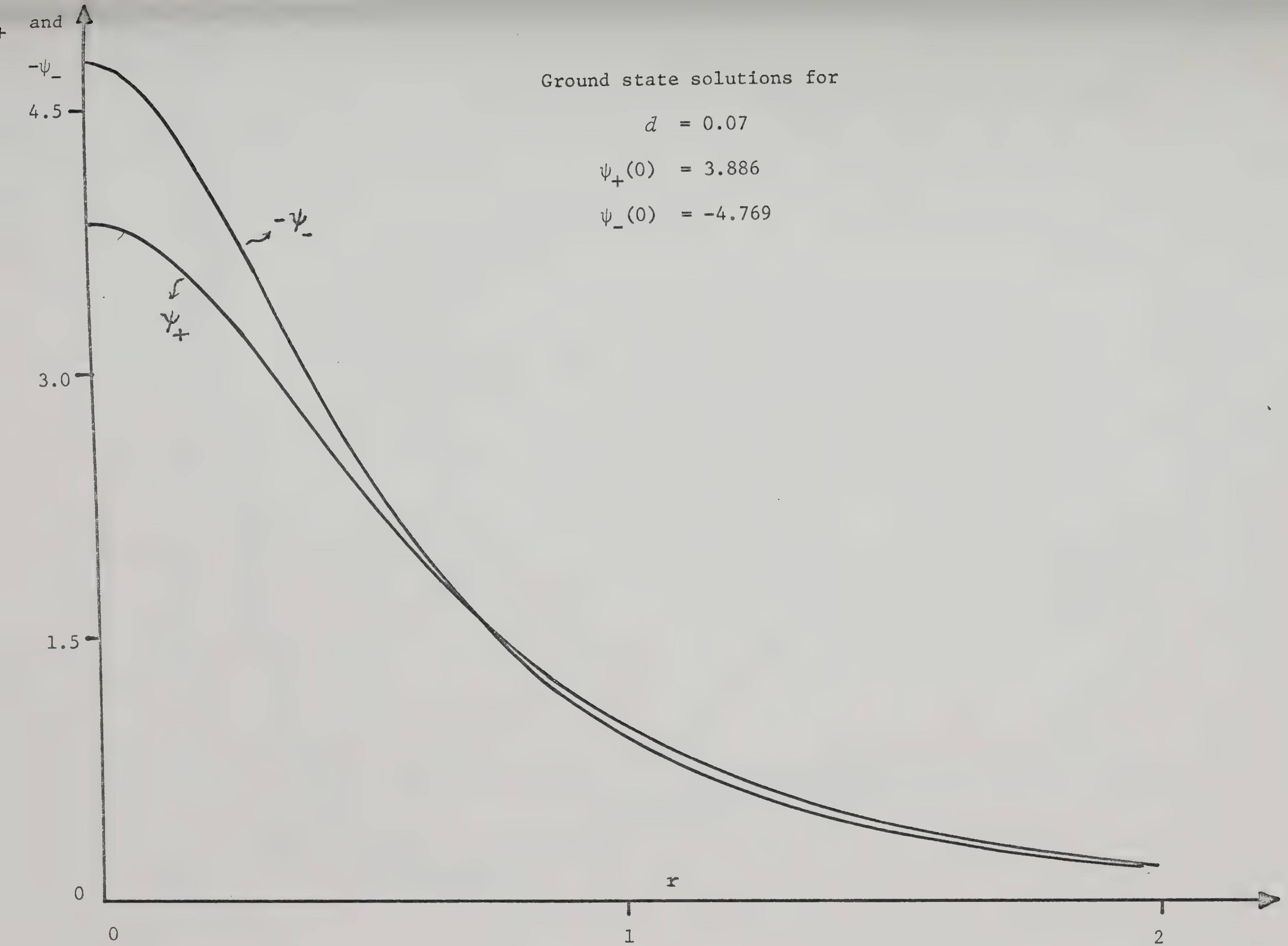

Ground State Solution
for $\alpha = 0$
$\psi_+(0) = 4.3339$
$\psi_+ = -\psi_-$
4.5
3.0
1.5
r
1
2

Fig. IV - 1



and
$-\psi_-$
4.5
3.0
1.5
0
0
1
2
r
$-\psi_-$
ψ_+
Ground state solutions for
$d = 0.07$
$\psi_+(0) = 3.886$
$\psi_-(0) = -4.769$

Fig. VI - 2

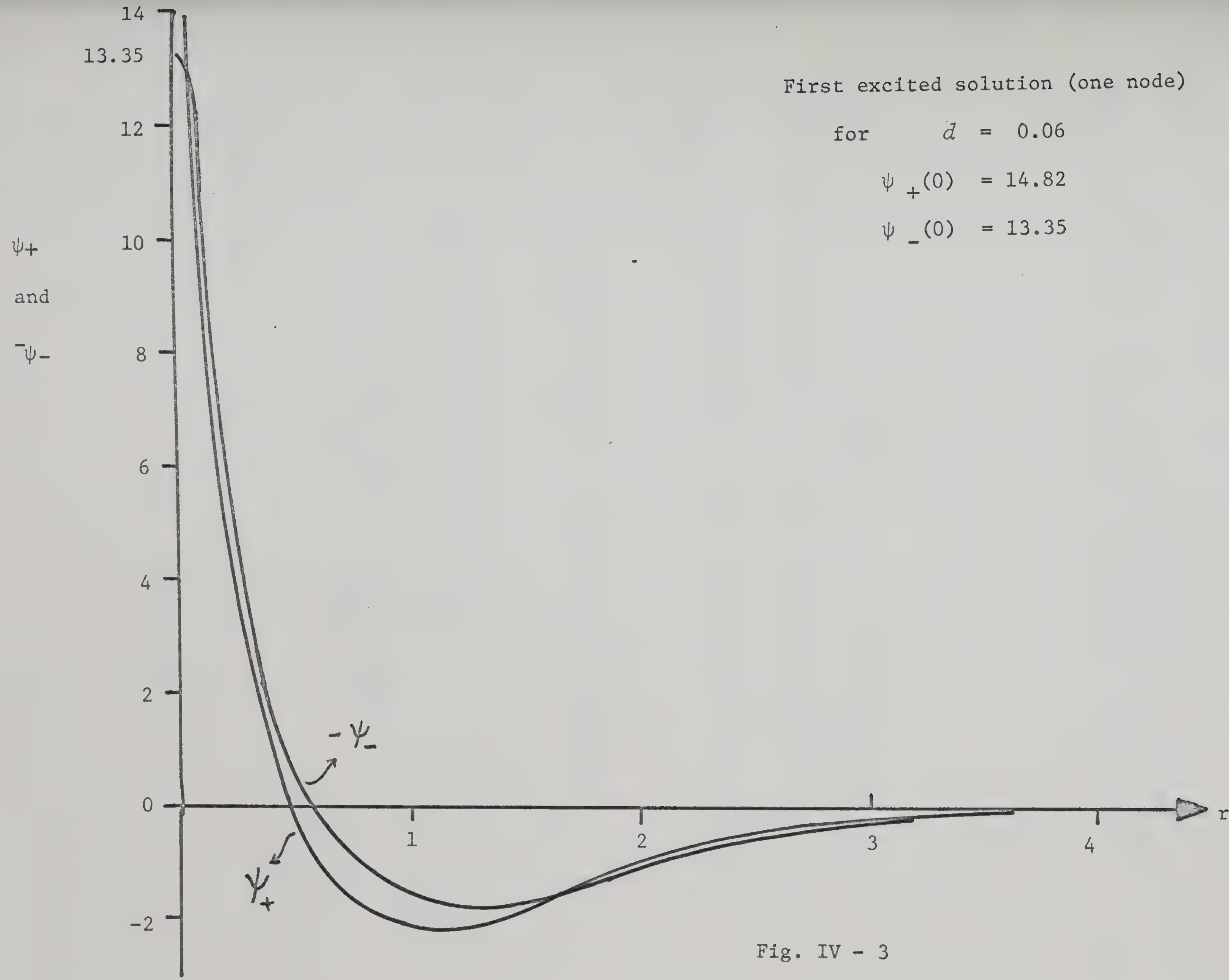

First excited solution (one node)
for d = 0.06
$\psi_+(0)$ = 14.82
$\psi_-(0)$ = 13.35
ψ_+ and $-\psi_-$
14
13.35
12
10
8
6
4
2
0
-2
1
2
3
4
r
$-\psi_-$
ψ_+

Fig. IV - 3

V. VARIATIONAL CALCULATIONS FOR NEUTRAL PARTICLES

First we study the spherically symmetric ground states solutions of (IV-3) variationally. Consider the simplest trial function

$$\psi = A\, e^{-\alpha r} \tag{V-1}$$

where A and α are the parameters (of amplitude and scale respectively) with respect to which we extremise the Lagrangian.

The Lagrangian for equation (IV-3) is (for spherically symmetric case)

$$L = \int \left[\tfrac{1}{2}(d\psi/dr)^2 - \tfrac{1}{4}\psi^4 - d\psi^3 + \frac{(1 - 3d^2)}{2}\psi^2\right] r^2\, dr \quad .$$

Using (V-1) and integrating we get

$$L = \frac{A}{8\alpha} + \frac{(1 - 3d^2)A^2}{8\alpha^3} - \frac{A^4}{128\alpha^3} - \frac{2d\, A^3}{27\alpha^3} \quad .$$

Then by setting

$$\partial L/\partial A = 0 \qquad \text{and} \qquad \partial L/\partial \alpha = 0$$

we have, to the 1st order in d

$$A_{\pm} = -\frac{64}{9} d \pm 4(2)^{1/2}$$

$$A_{\pm}^{2} = 32 \mp \frac{512}{9} (2)^{1/2} d$$

$$\alpha_{\pm}^{2} = 3 \mp \frac{32}{9} (2)^{1/2} d$$

$$L_{\pm} = 1.53 \mp 2.5\, d \quad .$$

We observe that trial function (V-1) does not satisfy $\psi'(0) = 0$. This however is not a drawback insofar as mass calculation is concerned. If desired the requirement $\psi'(0) = 0$ can be included as a subsidiary condition in the variational procedure. To this end another function was tried,

$$\psi = A(\alpha r + 1)e^{-\alpha r}$$

which satisfies $\psi'(0) = 0$, identically.

This trial function gave

$$A_{\pm} = -5.277\, d \pm 4.05$$

$$A_{\pm}^{2} = 16.4027\, (1 \mp 2.64\, d)$$

$$\alpha_{\pm}^{2} = 7.0049\ (1 \mp 7.0\ d)$$

$$L_{\pm} = 1.55 \mp 51.6\ d$$

to the 1st order in d .

Results of the variational calculations are tabulated along with the numerical calculation (for $d = 0$) for comparison. The values of L (= mass, M) obtained variationally are within 2% and 3.4% of the numerical value. The additional condition $\psi'(0) = 0$ imposed on the second trial function gives a better $\psi(0)$ (=A) but raises the minimum. A better fit would be obtained if we choose a trial function which behaves, in the asymptotic region, as $e^{-\alpha r}/r$.

Table (V - 1)

Trial Function	ψ (0) (= A)	Mass (=L)
$A\ e^{-\alpha r}$	$\pm\ 5.64 - 7.1\ d$	$1.53 \mp 2.5\ d$
$A(\alpha r + 1)e^{-\alpha r}$	$\pm\ 4.05 - 5.28\ d$	$1.55 \mp 51.6\ d$
Numerical result ($d = 0$)	$\pm\ 4.339$	1.503788

It may be pointed out that integral relations (IV-8) and (IV-9) are equivalent to the extremisation of L through amplitude

variation and scale variation respectively, and may be used to determine A and α .

Axially symmetric solutions were studied. From the nature of the equation satisfied by ψ it could be seen that odd parity solutions are to be excluded. The following even parity trial functions were considered.

1. $\psi = A\alpha^2 r^2 e^{-\alpha r} \cos^2\theta$

2. $\psi = A\alpha^2 r^2 e^{-\alpha r} P_2(\cos\theta)$

3. $\psi = A\, e^{-\alpha r} [1 + \alpha^2 r^2 B\, P_2(\cos\theta)]$.

Parameters A, α, and B were obtained to the 1st order in d . A mixed parity function of the type

$$\psi = A\, e^{-\alpha r} [\alpha\, r\, P_1(\cos\theta) + \alpha^2 r^2 B\, P_2(\cos\theta)]$$

was also tried. In the small d limit it lead to an odd parity function $(B = 0)$.

The results for the three functions above are tabulated on the next page.

Trial Function	A	α	B	L
$\psi = A\alpha^2 r^2 e^{-\alpha r}\cos^2\theta$	$\pm 4.614 - 4.567\,d$	$2.323 \pm 1.394\,d$	-	$1.908\,(1 \mp 1.38\,d)$
$\psi = A\alpha^2 r^2 e^{-\alpha r} P_2(\cos\theta)$	$\pm 2.98 - 1.098\,d$	$\sqrt{3}\,(1 \mp 0.246\,d)$	-	$1.905\,(1 \mp 0.497\,d)$
$\psi = Ae^{-\alpha r}[\,1 + \alpha^2 r^2 B P_2(\cos\theta)]$	$\pm 0.913 - 0.752\,d$	$1.732\,.(1 \mp 0.549\,d)$	9.885	$1.915\,(1 \mp 1.552\,d)$

Table(V - 2)

VI. CHARGED PARTICLES

In the previous chapter, it was seen that for $\overline{A} = 0$ solutions were obtained representing neutral particles. Now with regard to further possible solutions of (IV-1) and (IV-2) we may consider the following situation:

$$\overline{A} = 0 \qquad \text{in some volume } V \text{ near the origin} \tag{VI-1}$$

and

$$A_i^2 + 1 = 0 \qquad \text{outside } V \tag{VI-2}$$

consequently equation (IV-2) becomes

$$\nabla^2\phi = \phi - \phi^3 + \rho \qquad \text{inside } V \tag{VI-3}$$

and

$$\nabla^2\phi = \rho \qquad \text{outside } V \tag{VI-4}$$

Since both (VI-1) and (VI-2) must hold on the surface S enclosing the volume V, we get a boundary condition

$$\phi = \pm 1 \qquad \text{on } S \,. \tag{VI-5}$$

Now the most general spherically symmetric solution of (VI-4) is

$$\phi_{out} = a + \frac{b}{r} + \frac{\rho}{6} r^2 \tag{VI-6}$$

where the constants a,b and ρ are, by boundary condition (VI-5), required to satisfy

$$a + \frac{b}{R} + \frac{\rho}{6} R^2 = \pm 1$$

where R is the radius of the spherical region V .

The expression (VI-6) represents a situation where there is a charge 'b' at the origin surrounded by a uniform charge density ρ .

Now as before, we have a continuum of well behaved solutions characterized by, say $\phi(0)$, for each group (M,N) where M and N are the nodes on the lines $\phi = 0$ and $\phi = \pm 1$. Hence in order to choose the discrete set of particle-like solutions we choose to invoke the action principle: only those solutions for which the total Lagrangian L is a minimum w.r.t. the parameters characterizing the continuum, will be considered to represent charged particles.

The first variation of L is given by

$$\delta L = \frac{1}{4\pi} \int_{S_\infty} \delta\phi \, \overline{\nabla}\phi \cdot dS$$ [1]

where S_∞ is a spherical surface of radius $R \to \infty$. With $\phi(0)$ as a parameter and 'a' and 'b' as functions of $\phi(0)$, using (VI-6) for ϕ_{out} we have

$$\delta L = \underset{r\to\infty}{\mathrm{Lt}} \left[\left(-b + \frac{\rho}{3} r^3\right)\delta a - \frac{\rho}{3} r^2 \, \delta b \right] . \qquad \text{(VI-7)}$$

Thus L will have an extremum only if 'a' and 'b' have simultaneous extrema. Numerical calculations for various values of $\phi(0)$ over a large range, for given values of ρ, did not exhibit such simultaneous extrema. We do not have, however, a general proof that such extrema do not exist.

It was seen in chapter III that, because $J = (0,0,0,i\rho)$ where ρ is a constant, we get a cannonical energy-momentum tensor (III-6) satisfying $T_{ik,k} = 0$. In general, this tensor is not symmetric and need not lead to the global conservation of energy-momentum. If we could symmetrise this tensor, preserving $T_{ik,k} = 0$, we can construct the angular momentum density tensor, say $M_{ik\ell}$ antisymmetric in i and k, which again may not give a global conservation, only a local conservation that is to say $M_{ik\ell,\ell} = 0$. Now even if we do not have global conservation we would like to see if we can have local conservation and

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1. G.W. Darewych, Ph.D. Thesis, Chapter V, p. 35.

may attempt to construct a symmetric energy-momentum tensor satisfying $T_{ik,k} = 0$. As seen before, expression (IV-4), the usual addition of a divergenceless term $(A_i F_{k\ell})_{,\ell}$ does not yield a symmetric tensor, except when $\overline{A} = 0$. We can construct a symmetric tensor using the definition[2]

$$\frac{1}{2}\sqrt{-g}\, T_{ik} = \frac{\partial}{\partial x^i} \frac{\partial \sqrt{-g}\, L}{\partial \frac{\partial g^{ik}}{\partial x^\ell}} - \frac{\partial \sqrt{-g}\, L}{\partial g^{ik}} \qquad \text{(VI-8)}$$

which also should ensure that $T_{ik,k} = 0$ (for Cartesian co-ordinates).

Then expressing the Lagrangian density (L) as

$$L = \frac{-N}{4\pi}\left(\frac{1}{4} F_{\ell m} F_{rs}\, g^{r\ell} g^{sm} + \frac{1}{2} A_\ell A_m\, g^{m\ell} + \frac{1}{4} A_\ell A_r A_m A_s\, g^{r\ell} g^{sm} + J_\ell A_m g^{m\ell} + K\right) \qquad \text{(VI-9)}$$

we get in the Cartesian system

$$T_{ik} = \frac{-N}{4\pi}\Big[\left(\frac{1}{4} F_{\ell m}^2 + \frac{1}{2} A_\ell^2 + \frac{1}{4} A_\ell^2 A_m^2 + J_\ell A_\ell + K\right)\delta_{ik} - F_{ij} F_{kj} - (1 + A_j^2)\, A_i A_k - (A_i J_k + A_k J_i)\Big] \qquad \text{(VI-10)}$$

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2. L.D. Landau and E.M. Lifschitz, The Classical Theory of Fields (revised 2nd edition), eqn. 94.4, p. 312.

But, as it is, this tensor, though symmetric, does not satisfy[3]

$$T_{ik,k} = 0 \quad .$$

If we require T_{ik} (VI-10) to satisfy above relation we need to impose an additional condition

$$\overline{\nabla} \cdot \overline{A} = 0 \quad .$$

And this condition would imply, for non-singular spherical symmetry, $\overline{A} = 0$ everywhere. Further, even with $\overline{A} = 0$, this tensor does not show particle like properties. The choice of the constant K which gives finite $\int T_{44}\, dV$, results in the divergence of $\int T_{ii}\, dV^3$. (For a particle we require $\int T_{11}\, dV = \int T_{22}\, dV = \int T_{33}\, dV = 0$.)

Hence it was thoughtthat, taking into account the contravariant character of the current, in the expression (VI-8) the term involving the back-ground current should be written as $J^{\ell}A^{r}\, g_{\ell r}$.[4] As a result we get

$$T_{ik} = \frac{-N}{4\pi} \left[\left(\frac{1}{4} F^2_{\ell m} + \frac{1}{2} A^2_{\ell} + \frac{1}{4} A^2_{\ell} A^2_m + J_{\ell} A_{\ell} + K \right) \delta_{ik} - F_{ij} F_{kj} - (1 + A^2_j) A_i A_k \right] \qquad \text{(VI-11)}$$

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3. See Appendix B, Section IIÍ.
4. I would like to thank Dr. Israel for the discussion on this point.

which is symmetric and satisfies

$$T_{ik,k} = 0 .$$

Since expression (VI-11) is arrived at without restrictions on A_i' , it is natural to expect that this tensor should give all the results of chapter IV wherein we obtained particle-like solutions with $\overline{A} = 0$ and $\psi = \phi' - d$. But actually, this tensor does not exhibit particle-like properties for such a special case. The choice of the constant K which gives, in the rest frame, finite $\int T_{44}\, dV$ results in the divergence of $\int T_{ii}\, dV$,[5] while for a particle we require, in the rest frame, $\int T_{11}\, dV = \int T_{22}\, dV = \int T_{33}\, dV = 0$.

Thus in constructing the energy-momentum tensor, we get finite particle-like transformations for a special case $\overline{A} = 0$ provided we do not insist on the symmetry of the general expression for T_{ik} . While if we insist on the symmetry (or local conservation of angular momentum) for most general case then for above special case we are faced with infinite integral.

Besides the construction of the energy-momentum tensor, the problem of separating out the charged particle from the back-ground remains. That is to remove the back-ground in such a way that one is left with a ϕ_{out} (particle) $= a_1 + \frac{b}{r}$ and a T_{ik} having particle-

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5. See Appendix B, Section IV.

like properties. It does not seem possible to do so in a relativistically invariant manner.

Such being the case, we may say that the system does not admit static charged particle-like solutions.

We, thus, seem to be unable to study the mass spectrum of the charged particles. But we may still attempt to establish the charge spectrum ('b') . As a heuristic approach, we may choose to consider those solutions for which 'a' (VI-7) has an extremum, since in that case the leading divergent term in L is reduced to r^2 from r^3 .

Equation (VI-3) was solved numerically using Runge-Kutta method. Constants 'a' and 'b' were determined by the boundary condition $\phi = \pm 1$ on S and the continuity of ϕ' .

The numerical results thus obtained are tabulated on the next page. Only (M = 1, N = 1) group was considered. As is obvious the difference between b_+ and b_- is noticeable for large values of d only.

Table (VI - 1)

1	a	b	$\phi(0)$
0	-2.177 +2.177	+2.760866 -2.760866	+6.98 -6.98
0.1	-2.074 +2.264	+2.984 -2.624	+6.4711 -7.38
0.2	-1.948 +2.336	+3.503 -2.539	+5.71 -7.69

a and $\phi(0)$ are in units of mc^2/g and b is in units of $\hbar c/g$.

VII. CONCLUSIONS

We have studied the effect of introducing a uniform constant charge density as a simple back-ground, on the non-linear field equations

$$F_{ik,k} = -(1 + A_k^2)\, A_i \quad .$$

New field equations proposed were

$$F_{ik,k} = -(1 + A_k^2)\, A_i - J_i \quad .$$

$J_i = (0,0,0,i\rho)$ being the back-ground current and ρ being a constant.

In a special case $\overline{A} = 0$ everywhere, it is possible to obtain a symmetric energy-momentum tensor with a local conservation law for the total system. Not only that, but by simply expressing the total potential ϕ as $\phi = \psi + d$, and a proper choice of d and K (the constant appearing in the Lagrangian density) leads us to proper energy-momentum tensor, in terms of ψ, having particle-like transformations. It was pointed out that the spherically symmetric solutions for ψ asymptotic to zero as $e^{-\alpha r}/r$ form a discrete set and represent neutral particles. Asymptotically, then, the total potential appears as an exponentially decaying contribution plus a constant back-ground term. Both these combine in the formulation of the energy-momentum tensor to yield a total system that has finite particle-like transformations. Thus the

back-ground is incorporated into the particle.

In a case where $\overline{A} = 0$ in some spherical region V near the origin and $A_i^2 = -1$ outside V, the expression for ϕ_{out}, in fact, represents a situation wherein a charge 'b' at the origin is surrounded by the constant uniform charge density ρ. But as seen in Chapter VI, because of the difficulties in the construction of the proper energy-momentum tensor, we are not able to study the mass spectrum (in fact one can say that we do not have any particle since we do not have proper energy-momentum tensor). Also the charge spectrum could be established only in a heuristic way.

As seen in Chapter IV the introduction of the back-ground results in the removal of the mass degeneracy between ψ_+ and ψ_- neutral states. Is it possible that in a more physical model such a removal of mass degeneracy would explain the small mass difference, say, between neutral K_1^o and K_2^o ? Since we are already on the speculative plane, it may be of interest to obtain some estimate of d (which gives a fractional mass difference comparable to that between K_1^o and K_2^o), the fractional charge difference between b_+ and b_- for this value of d and the corresponding back-ground charge density ρ. As can be seen from the graph C - 1 (Appendix C), $d \approx 2.5 \times 10^{-15}$ gives us the required mass difference between M_+ and M_- for the first excited state. (In this theory of Bosons it is natural to identify the nodeless ground state with the Pions.). Further, graph C - 2 (Appendix C) shows that for this value of d the fractional charge difference between b_+ and b_- is of the order 8×10^{-18}. Of course this could only be considered as a rough estimate (obtained

by extrapolation). It is interesting though to note that this estimate of the fractional charge difference compares closely with the estimate of the fractional charge difference between proton and electron given by H. Bondi (Chapt. I ref. 4). H. Bondi, wherein he uses the charge inequality of proton and electron to explain the expanding universe, obtains a value of the order 2×10^{-18}. R.W. Stover et. al. (Chapt. I ref. 4) found in their experiment an upper limit for the fractional charge difference between proton and electron, which is of the order 0.8×10^{-19}. It may be stressed here that all our estimates of d and the fractional charge difference are independent of the universal constants m and g (appearing in eqn. III - 1).

Further to obtain an estimate of $\rho = (d^3 - d).g/(\hbar/mc)^3$ we need to determine g and m. To this end if we identify the charge b with the electronic charge (i.e. $b\hbar c/g \approx e$), we get $g = 137\ be \approx 1.7\times10^{-7}$ e.s.u. Similarly setting the mass of the 1st excited state equal to that of K^o we get $m \approx 6\times10^{-22}$ gm. As a result $\rho \approx 1.4\times10^{27}$ e.s.u./c.c.

Now if we look upon the back-ground simply as arising out of the charge difference between the positive and negative charges, the value of ρ obtained here is not consistent with the known density of the charged particles in the universe. Again, to what extent then the back-ground charge density can be considered as a vacuum property is not known, however our results clearly show that some such consideration would have to be introduced.

The investigation of plane wave-like solutions proved unsuccesful so far, except for $A_k^2 = -1$ which leads us to the Maxwellian linear theory.

Though dynamical stability of the neutral particle-like solutions was not studied, it is obvious that they are unstable in the sense of Derrick's theorem (Chapter II, p. 8), since they are the solutions of the equation of the type $\nabla^2\phi = F'(\phi)$.

APPENDIX A

PHASE SPACE ANALYSIS

Consider the equation for ψ

$$\nabla^2\psi = (1 - 3d^2)\psi - 3d\,\psi^2 - \psi^3 .$$

For the spherical symmetry, we may write

$$\psi = y(r) .$$

Then equation for y is

$$y'' + \frac{2}{r}y' = (1 - 3d^2)y - 3d\,y^2 - y^3$$

and we assign, at the origin $y' = 0$.

The equation has trivial special solution

$$y = \text{constant} = C \quad \text{say}$$

then

$$C = 0$$

and

$$C = C_{\pm} = -\frac{3}{2}d \pm \left(1 - \frac{3}{4}d^2\right)^{1/2}$$

$$\left(\text{for real } C_{\pm} ,\ |d| < 2/(3)^{1/2}\right) .$$

Consider the solution in the neighbourhood of special solutions

$$y = C + u \quad .$$

Then in asymptotic region, for small u, linearised equations satisfied by u are

$$u'' + \frac{2}{r} u = (1 - 3d^2)\, u \qquad \text{near} \quad C = 0$$

$$= -(2 \mp 3d)\, u \qquad \text{near} \quad C_{\pm}$$

The solutions are, in general, spherical Bessel functions. We want the eigen-solutions (near C = 0) to approach the axis exponentially in order to have quadratic integrability. This imposes a condition

$$|d| < 1/(3)^{1/2} \quad .$$

Thus in the asymptotic region, we have

$$u \to \frac{1}{r} \cdot e^{-(1 - 3d^2)^{1/2}\, r} \qquad \text{near} \quad C = 0$$

and

$$u \to \frac{1}{r} \cdot \sin\, (2 \mp 3d)^{1/2}\, r \qquad \text{near} \quad C_{\pm}$$

The different behavior near $C_{\pm}$ and the axis correspond to the different nature of these special solutions, as will be seen on the phase-plane.

Now if we consider y and y' as position and velocity of a representative point, then given equation describes a non-conservative motion. The energy for the corresponding conservative motion is

$$K = \frac{1}{2}(y')^2 + V(y)$$

where

$$V(y) = \frac{1}{4}y^4 + d\,y^3 - \frac{(1 - 3d^2)}{2}y^2 \; .$$

The equilibrium points of this motion are given by $\partial V/\partial y = 0$ and correspond to the special solutions $y = 0$ and $y = C_{\pm}$.

Figure (A-1) represents the situation for $d = 0$, while Figure (A-2) represents the situation for $d = 0.1$. A point representative of the conservative motion will move on the curves of constant K , while for actual motion (non-conservative) we may consider

$$d\,K/dr = -2(y')^2/r$$

which is always negative. Thus a representative point of the actual

motion will always move inwards across the lines of constant K. Such a trajectory must always terminate either at $C_{\pm}$ or the origin.

For initial value of y such that $0 < y < z$, the trajectory will always terminate at C_+. If we increase initial y, slightly, to say y_1 the trajectory may still terminate at C_+. If we further increase it to say y_2 it may terminate at C_-, while for some value $\bar{y}(y_1 < \bar{y} < y_2)$ it will terminate at the origin. The trajectory $\bar{y}0$ represents the eigen-solution. Solutions corresponding to the trajectories y_1C_+, $\bar{y}0$ and y_2C_- are described schematically in Figure (A-3).

By narrowing the interval $y_1 - y_2$ the eigen-solutions may be determined with arbitrary accuracy.

By increasing initial value of y further we can repeat the same proceedure to obtain a discrete set of eigen-solutions.

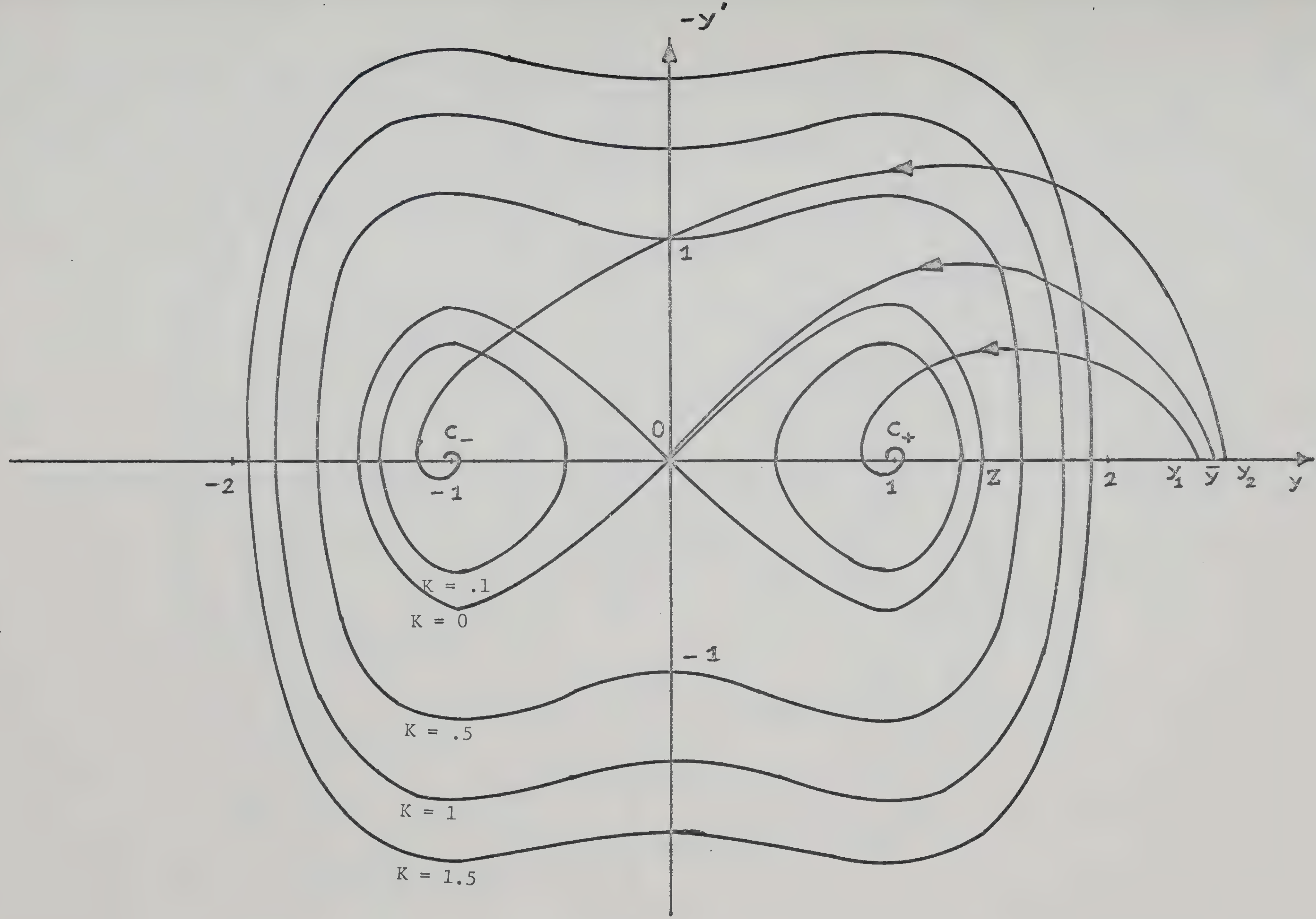


Fig. A - 2

$d = 0$

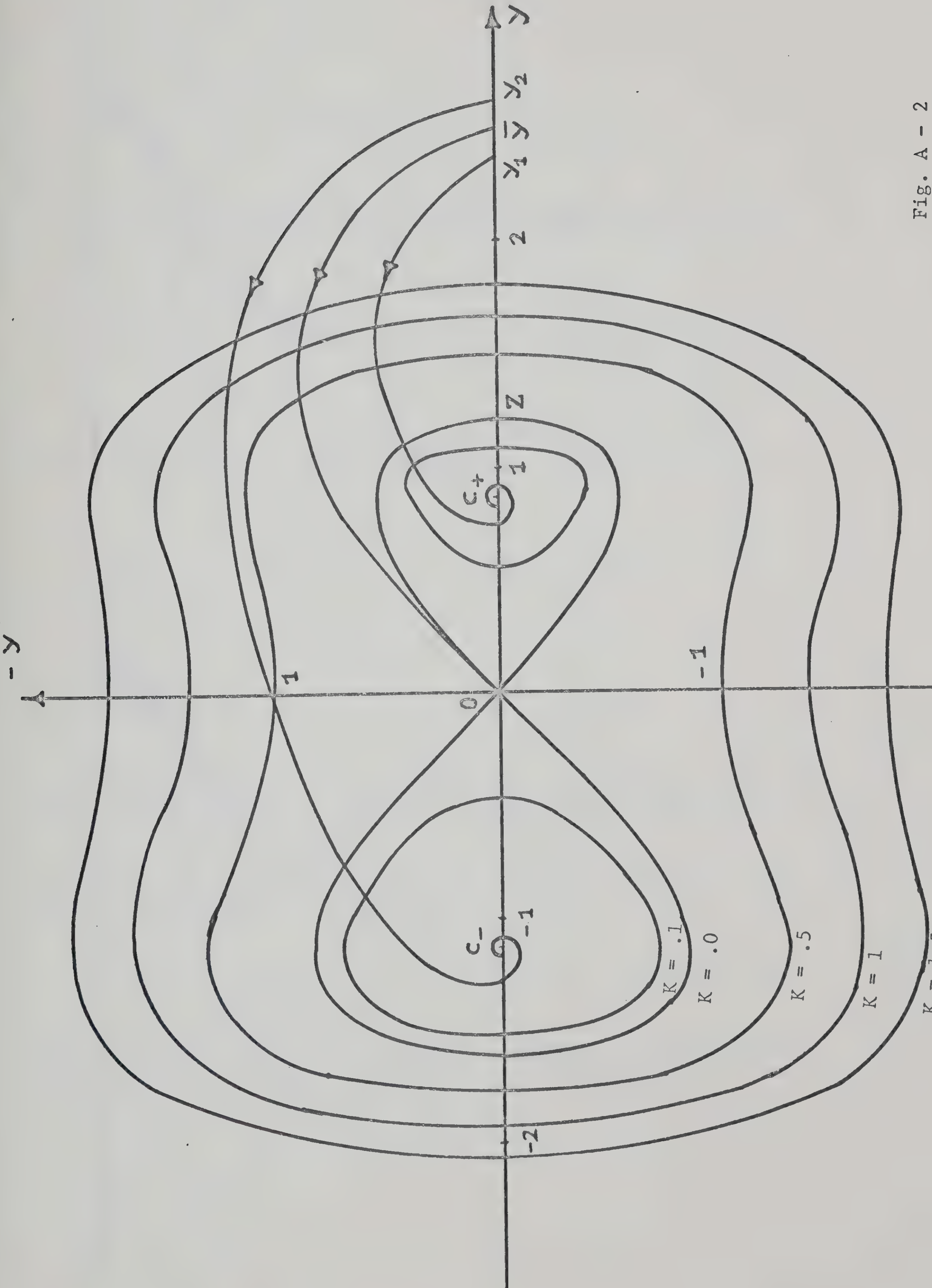


Fig. A - 2

$d = 0.1$

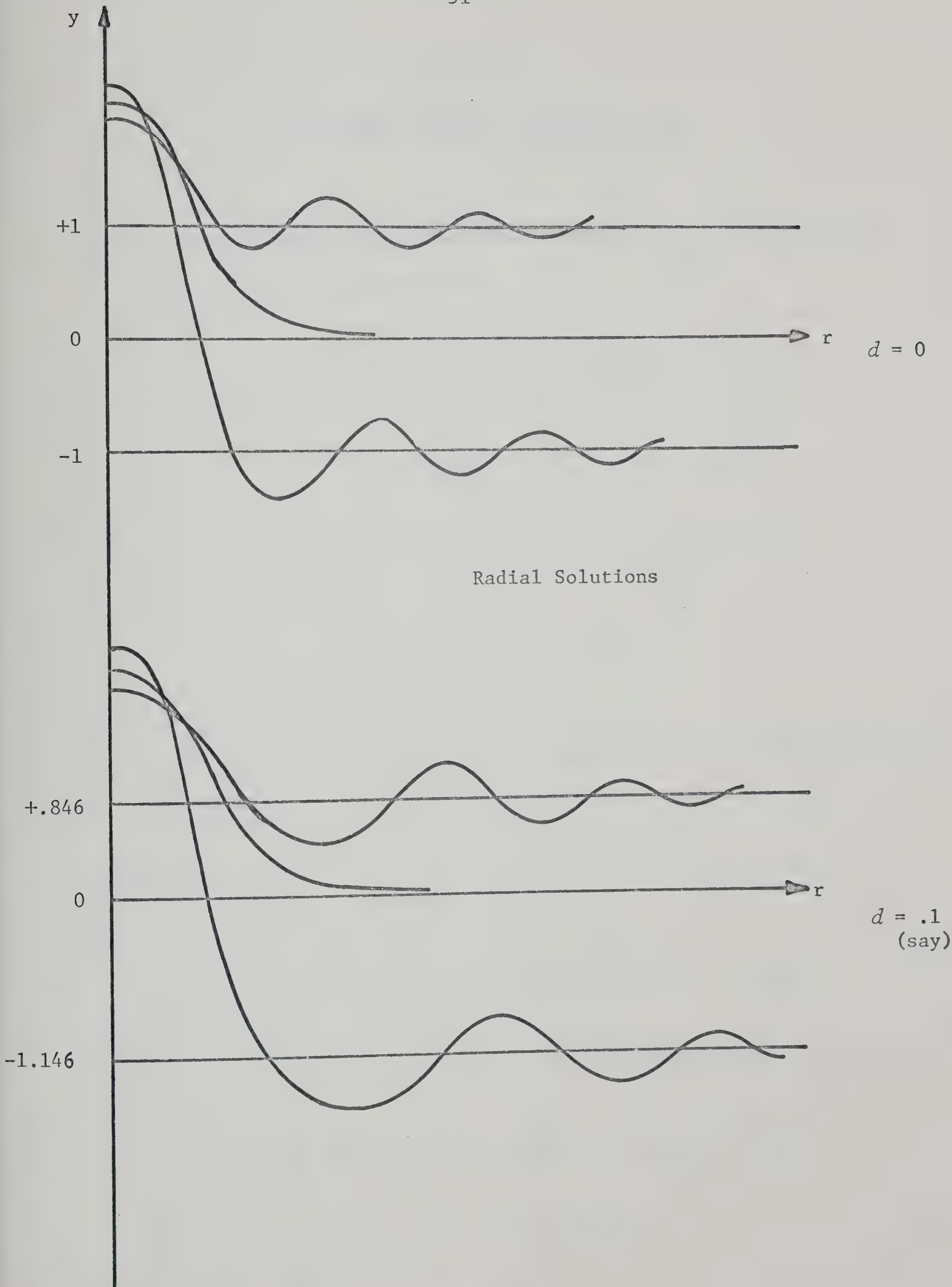


Fig. A - 3

APPENDIX B

CONCERNING ENERGY MOMENTUM TENSOR

I. The cannonical energy-momentum tensor T_{ik} for the system is given by (III-7). It satisfies

$$T_{ik,k} = \frac{1}{4\pi} [\frac{1}{2} F_{\ell m}F_{\ell m,i} + A_{\ell}A_{\ell,i} + A_m^2 A_{\ell}A_{\ell,i} + A_{\ell,i} J_{\ell}$$

$$+ A_{\ell} J_{\ell,i} - A_{j,i,k} F_{kj} - A_{j,i} F_{kj,k}]$$

$$= \frac{1}{4\pi} [[A_{\ell}(1 + A_m^2) + J_{\ell}]A_{\ell,i} - A_{\ell,i} F_{m\ell,m}$$

$$+ \frac{1}{2} F_{\ell m} F_{\ell m,i} - A_{\ell,i,m} F_{m\ell}] \ .$$

$$= 0$$

Using field equations, definitions of $F_{\ell m}$ and $J_i = (0,0,0,i\rho)$, where ρ is a constant.

Consider

$$T_{ik,i} = \frac{1}{4\pi} [\frac{1}{2} F_{\ell m} F_{\ell m,k} + A_{\ell}A_{\ell,k} + A_m^2 A_{\ell}A_{\ell,k} + A_{\ell,k} J_{\ell}$$

$$+ A_{\ell}J_{\ell,k} - A_{j,i,k} F_{kj} - A_{j,i} F_{kj,i}]$$

$$= \frac{1}{4\pi} [\frac{1}{2} F_{\ell m} F_{\ell m,k} + A_{\ell}A_{\ell,k} + A_m^2 A_{\ell}A_{\ell,k} + A_{\ell,k} J_{\ell}$$

$$- A_{\ell,m,m} F_{k\ell} - A_{\ell,m} F_{k\ell,m}] \ .$$

Thus,

$$T_{ik,i} \neq 0 \qquad\qquad \text{in general}$$

but for a static case with $A_\alpha = 0$, $\alpha = 1,2,3$ we get

$$T_{i4,i} = \frac{1}{4\pi} [\frac{1}{2} F_{\ell m}F_{\ell m,4} + A_\ell A_{\ell,4} + A_m^2 A_\ell A_{\ell,4}$$

$$+ A_{\ell,4} J_\ell - A_{\ell,m,m} F_{4\ell} - A_{\ell,m} F_{4\ell,m}]$$

$$= \frac{1}{4\pi} [-A_{4,m,m} F_{44} - A_{4,m} F_{44,m}] \qquad (\because A_\alpha = 0)$$

$$= 0$$

also

$$T_{i\alpha,i} = \frac{1}{4\pi} [\frac{1}{2} F_{\ell m} F_{\ell m,\alpha} + A_\ell A_{\ell,\alpha} + A_m^2 A_\ell A_{\ell,\alpha}$$

$$+ A_{\ell,\alpha} J_\ell - A_{\ell,m,m} F_{\alpha\ell} - A_{\ell,m} F_{\alpha\ell,m}]$$

$$= \frac{1}{4\pi} [-\phi_{,\beta} \phi_{,\beta,\alpha} - \phi\phi_{,\alpha} + \phi^2 \cdot \phi \cdot \phi_{,\alpha}$$

$$- \rho\phi_{,\alpha} + \phi_{,\beta,\beta} \phi_{,\alpha} + \phi_{,\beta}\phi_{,\alpha,\beta}]$$

$$= 0 \qquad [\because \phi_{,\beta,\beta} = \nabla^2\phi = \phi - \phi^3 + \rho]$$

In fact, as commented in Chapter IV, p. (19), for this special case T_{ik} is symmetrion. To see this let us examine the term $A_{j,i}\, F_{kj}$. The only non-vanishing contribution comes from $A_{4,i}\, F_{k4}$ and for static case this is of the type $\phi_{,\alpha}\phi_{,\beta}$.

Now consider

$$T_{44} = \frac{1}{4\pi}\,[\frac{1}{4}\,F^2_{\ell m} + \frac{1}{2}\,A^2_{\ell} + \frac{1}{4}\,A^2_{\ell}A^2_{m} + A_{\ell}\,J_{\ell} + K$$

$$- A_{j,4}\,F_{4j}] \quad .$$

For a static case with $\overline{A} = 0$

$$T_{44} = -\frac{1}{2}\,(\nabla\phi)^2 - \frac{1}{2}\,\phi^2 + \frac{1}{4}\,\phi^4 - \rho\phi + K$$

$$= -\frac{1}{2}\,(\nabla\psi)^2 - \frac{1}{2}\,\psi^2 - \frac{1}{2}\,d^2 - d\,\psi + \frac{1}{4}\,(\psi^4 + d^4 + 6d^2\psi^2 + 4d^3\psi$$

$$+ 4d\psi^3) - \rho\psi - \rho d + K \qquad (\text{where } \phi = \psi + d)$$

$\therefore$ for finite $\int T_{44}\, dV$ we must choose

$$K = \frac{1}{2}\,d^2 - \frac{1}{4}\,d^4 + \rho d.$$

Further, with this K

$$T_{11} = \frac{1}{4\pi}\,[\frac{1}{4}\,F^2_{\ell m} + \frac{1}{2}\,A^2_\ell + \frac{1}{4}\,A^2_\ell A^2_m + A_\ell\,J_\ell + K - A_{j,1}\,F_{1j}]$$

$$= \frac{1}{4\pi}\,[-\,\frac{1}{2}\,(\nabla\phi)^2 - \frac{1}{2}\,\phi^2 + \frac{1}{4}\,\phi^4 - \rho\phi + K + (\frac{\partial\phi}{\partial x})^2]$$

$$= \frac{1}{4\pi}\,\{\frac{1}{2}\,[(\frac{\partial\psi}{\partial x})^2 - (\frac{\partial\psi}{\partial y})^2 - (\frac{\partial\psi}{\partial z})^2] - \frac{1}{2}\,\psi^2 - \frac{1}{2}\,d^2 - \psi d$$

$$+ \frac{1}{4}\,(\psi^4 + d^4 + 6d^2\psi^2 + 4d^3\psi + 4d\psi^3) - \rho\psi - \rho d$$

$$+ \frac{1}{2} d^2 - \frac{1}{4} d^4 + \rho d\,\}$$

$$\therefore \quad \int T_{11}\,dV = -\,\frac{1}{2}\int(\frac{\partial\psi}{\partial z})^2\,dV - \frac{1}{2}\int(i-3d^2)\psi^2\,dV$$

$$+ \frac{1}{4}\int \psi^4\,dV + \int d\;\psi^3\,dV$$

$$= 0$$

herein we have used the relations (IV-9) and (IV-11).

Similarly

$$\int T_{22}\,dV = 0 = \int T_{33}\,dV \quad ,$$

II. In order to symmetrize the energy-momentum tensor one may choose to add a divergenceless term $(A_i F_{k\ell})$, to the cannonical energy-momentum tensor. This leads us to the T_{ik} given by (IV-4). This tensor is symmetric when $\bar{A} = 0$ and $T_{ik,i} = 0$ follows from $T_{ik,k} = 0$.

Consider

$$T_{ik,k} = \frac{1}{4\pi}\,[\tfrac{1}{2} F_{\ell m}F_{\ell m,i} + A_\ell A_{\ell,i} + A_m^2 A_\ell A_{\ell,i} + A_{\ell,i} J_\ell + A_\ell J_{\ell,i}$$

$$- F_{ij,i} F_{kj} - F_{ij} F_{kj,j} - (1 + A_j^2) A_{i,k} A_k$$

$$- (1 + A_j^2) A_i A_{k,k} - 2A_j A_i A_k A_{j,k} - A_{i,k} J_k$$

$$- A_i J_{k,k}]$$

$$= \frac{1}{4\pi}\{[A_\ell(1 + A_m^2) + J_\ell] A_{\ell,i} - F_{i\ell} [(1 + A_m^2)A_\ell + J_\ell]$$

$$- [A_\ell(1 + A_m^2) + J_\ell] A_{i,\ell} - A_2[(1 + A_m^2)A_\ell]_{,\ell}\}$$

$= 0$ [last term drops out because of the gauge condition (III-5)]

Now consider

$$T_{44} = \frac{1}{4\pi} [\frac{1}{4} F_{\ell m}^2 + \frac{1}{2} A_\ell^2 + \frac{1}{4} A_\ell^2 A_m^2 + A_\ell J_\ell + K$$

$$- F_{4j}^2 - (1 + A_m^2) A_4 A_4 - A_4 J_4$$

$$= \frac{1}{4\pi} [\frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} \phi^2 - \frac{3}{4} \phi^4 + K]$$

$$= \frac{1}{4\pi} [\frac{1}{2} (\nabla\psi)^2 + \frac{1}{2} \psi^2 + \frac{1}{2} d^2 + \psi d$$

$$- \frac{3}{4} (\psi^4 + d^4 + 6d^2\psi^2 + 4d^3\psi + 4d\psi^3) + K]$$

$$(\because \quad \phi \doteq \psi + d)$$

then for finite $\int T_{44}\, dV$

$$K = - \frac{1}{2} d^2 + \frac{3}{4} d^4 .$$

Further, with this K

$$T_{11} = \frac{1}{4\pi} [\frac{1}{4} F_{\ell m}^2 + \frac{1}{2} A_\ell^2 + \frac{1}{4} A_\ell^2 A_m^2 + A_\ell J_\ell + K - F_{1j}^2$$

$$- (1 + A_j^2) A_1 A_1 - A_1 J_1]$$

$$= \frac{1}{4\pi} [- \frac{1}{2} (\nabla\phi)^2 - \frac{1}{2} \phi^2 + \frac{1}{4} \phi^4 - \rho\phi + K + (\frac{\partial\phi}{\partial x})^2]$$

which is the same as in section I. Hence

$$\int T_{11}\, dV = 0 = \int T_{22}\, dV = \int T_{33}\, dV .$$

Thus the addition of the divergenceless term $(A_i F_{k\ell})_{,\ell}$ does not, really, give anything different from the cannonical energy-momentum tensor.

III. It was attempted to construct a symmetric energy-momentum tensor, with a local conservation law, using the definition (VI-8). The expression (VI-9) for L leads us to a tensor (VI-10) which, when required to give a local conservation, makes it neccessary to choose $\overline{A} = 0$ for non-singular spherically symmetric case.

For T_{ik} (VI-9)

$$\begin{aligned} T_{ik,k} = \frac{1}{4\pi} [\tfrac{1}{2} F_{\ell m} F_{\ell m,i} &+ A_\ell A_{\ell,i} + A_m^2 A_\ell A_{\ell,i} + A_{\ell,i} J_i + A_\ell J_{\ell,i} \\ &- F_{ij,k} F_{kj} - F_{ij} F_{kj,j} - (1 + A_j^2) A_{i,k} A_k \\ &- (1 + A_j^2) A_i A_{k,k} - 2A_j A_i A_k A_{j,k} \\ &- A_{i,k} J_k - A_i J_{k,k} - A_{k,k} J_i - A_k J_{i,k}] \\ = -A_{k,k} J_i \end{aligned}$$

[Using definition of F_{ik}, the field equations and the gauge condition.]

Thus if we require $T_{ik,k} = 0$, we must have $\nabla\cdot A = 0$, for static case.

With $\overline{A} = 0$ and $\phi = \psi + \ell$

$$T_{44} = \frac{1}{4\pi}\,[\frac{1}{4}\,F_{\ell m}^2 + \frac{1}{2}\,A_\ell^2 + \frac{1}{4}\,A_\ell^2\,A_m^2 + A_\ell\,J_\ell + K$$

$$- F_{4j}^2 - (1 + A_j^2)\,A_4A_4 - 2A_4J_4]$$

$$= \frac{1}{4\pi}\,[\frac{1}{2}\,(\nabla\phi)^2 + \frac{1}{2}\,\phi^2 - \frac{3}{4}\,\phi^4 + \rho\phi + K]$$

$$= \frac{1}{4\pi}\,[\frac{1}{2}\,(\nabla\psi)^2 + \frac{1}{2}\,\psi^2 + \psi d + \frac{1}{2}\,d^2 - \frac{3}{4}\,(\psi^4 + d^4 + 6d^2\,\psi^2$$

$$+ 4d^3\psi + 4d\psi^3) + \rho\psi + \rho d + K]$$

$\therefore$ for finite $\int T_{44}\,dV$

$$K = -\frac{1}{2}\,d^2 + \frac{3}{4}\,d^4 - \rho d$$

With this K

$$T_{ii} = \frac{1}{4\pi}\,[F_{\ell m}^2 + 2A_\ell^2 + A_\ell^2\,A_m^2 + 4A_\ell\,J_\ell + 4K$$

$$- F_{ij}^2 - (1 + A_i^2)\,A_i^2 - 2\,A_i\,J_i]$$

$$= \frac{1}{4\pi}\,[-\phi^2 - 2\rho\phi + 4K]$$

$$= \frac{1}{4\pi}\,[-\psi^2 - 2\psi d - d^2 - 2\rho\psi - 2\rho d - 2d^2 + 3d^4 - 4\rho d]$$

$$= \frac{1}{4\pi} [-\psi^2 - 2\psi d - 2\psi\rho - 3\rho d]$$

Thus $\int T_{ii}\, dV$ diverges.

IV. Consider the T_{ik} (VI-11) , which emphasises the contravareant nature of current.

$$T_{ik,k} = \frac{1}{4\pi} [\frac{1}{2} F_{\ell m} F_{\ell m,i} + A_\ell A_{\ell,i} + A_m^2 A_\ell A_{\ell,i} + A_{\ell,i} J_\ell$$

$$+ A_\ell J_{\ell,i} - F_{ij,k} F_{kj} - F_{ij} F_{kj,k}$$

$$- (1 + A_j^2) A_{i,k} A_k - (1 + A_j^2) A_i A_{k,k} - 2A_j A_i A_k A_{j,k}]$$

$$= - A_{i,k} J_k$$

$$= 0 \qquad \text{for static case.}$$

Now

$$T_{44} = \frac{1}{4\pi} [\frac{1}{4} F_{\ell m}^2 + \frac{1}{2} A_\ell^2 + \frac{1}{4} A_\ell^2 A_m^2 + A_\ell J_\ell + K$$

$$- F_{4j}^2 - (1 + A_j^2) A_4 A_4]$$

$$= \frac{1}{4\pi} [\frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} \phi^2 - \frac{3}{4} \phi^4 - \rho\phi + K] \quad [\text{for static case with } \overline{A} = 0]$$

$$= \frac{1}{4\pi} [\frac{1}{2} (\nabla\psi)^2 + \frac{1}{2} \psi^2 + \frac{1}{2} d^2 + \psi d - \frac{3}{4} (\psi^4 + d^4$$

$$+ 6d^2\psi^2 + 4d^3\psi + 4d\psi^3) - \rho\psi - \rho d + K]$$

$\therefore$ for finite $\int T_{44}\, dV$

$$K = - \frac{1}{2} d^2 + \frac{3}{4} d^4 + \rho d.$$

The with this K

$$T_{ii} = \frac{1}{4\pi} [F^2_{\ell m} + 2A^2_\ell + A^2_\ell A^2_m + 4A_\ell J_\ell + 4K$$

$$- F^2_{ij} - (1 + A^2_j) A^2_i]$$

$$= \frac{1}{4\pi} [-\phi^2 - 4\rho\phi + 4K]$$

$$= \frac{1}{4\pi} [-\psi^2 - 2\psi d - d^2 - 4\rho\psi + 3d^4 - 2d^2]$$

$$= \frac{1}{4\pi} [-\psi^2 - 2\psi(d+4\rho) - 3d^2 + 3d^4]$$

Hence $\int T_{ii}\, dV$ diverges.

APPENDIX C

Herein we use extrapolation in order to determine d which gives a fractional mass difference comparable to that between K_1^o and K_2^o. Further by the same method the fractional charge difference for this value of d is determined.

With $$\Delta M = \left|\frac{M_+}{M_-} - 1\right|$$

using numerically integrated solutions (Table IV - 2),

d	Δ M	-log ΔM	-log d
10^{-10}	6.82×10^{-10}	9.167	10.0
10^{-8}	6.82×10^{-8}	7.167	8.0
10^{-6}	6.82×10^{-6}	5.167	6.0
10^{-4}	6.83×10^{-4}	3.167	4.0
10^{-3}	6.84×10^{-3}	2.165	3.0

For K_1^o and K_2^o, $\Delta M_{K^o} \approx 1.7\times10^{-14}$.

Hence from the graph (C - 1), $d \approx 2.5\times10^{-15}$ gives a fractional mass difference of the order 1.7×10^{-14}.

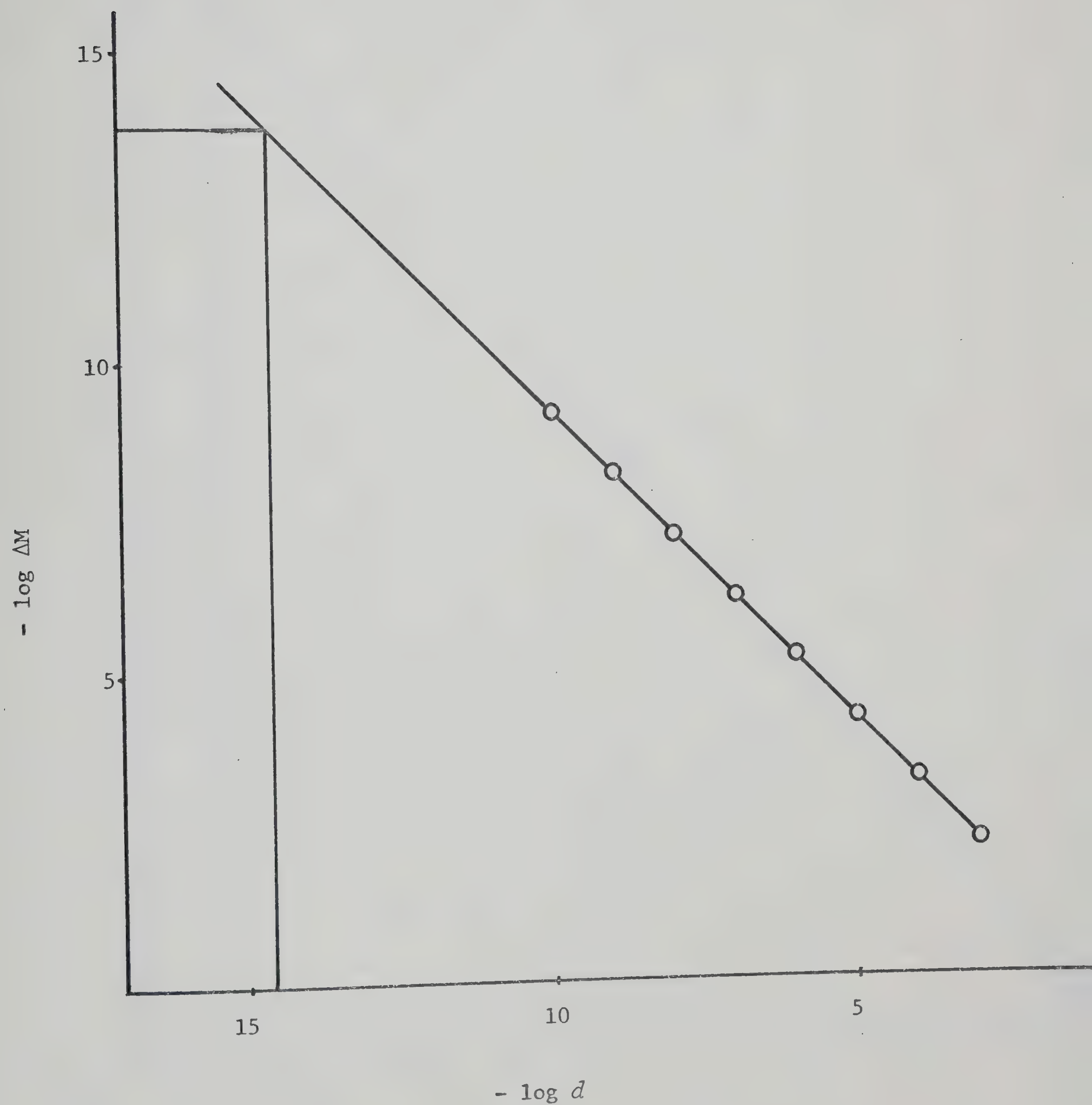
With $$\Delta b = \left|\frac{b_+}{b_-} - 1\right|$$

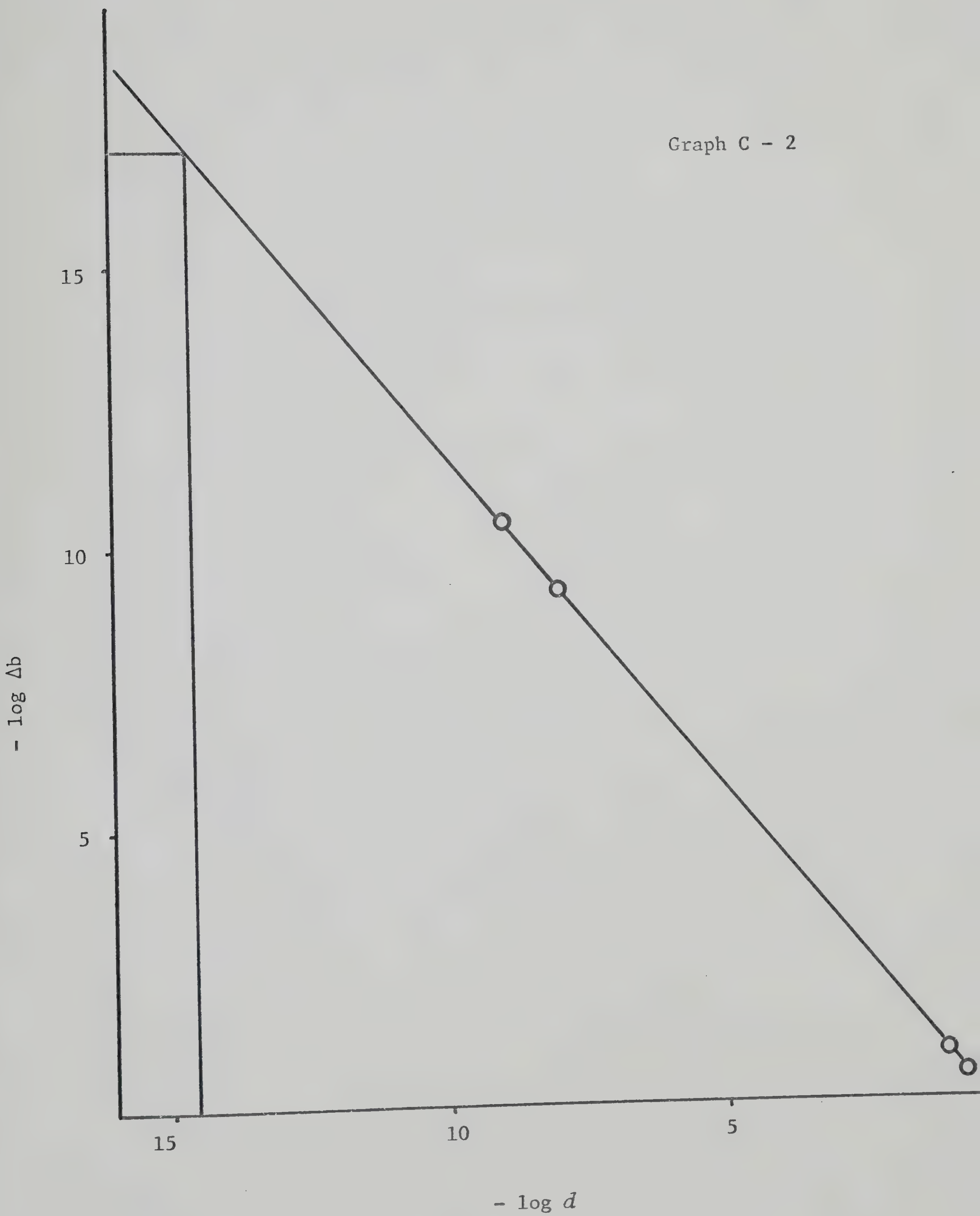
using numerically integrated solutions for which 'a' has an extremum

d	Δb	$-\log \Delta b$	$-\log d$
10^{-9}	4.0×10^{-11}	10.389	9.0
10^{-8}	6.9×10^{-10}	9.162	8.0
10^{-1}	0.137	0.863	1.0
2×10^{-1}	0.38	0.463	0.69

From the graph (C - 2), $d \approx 2.5\times10^{-15}$ gives $\Delta b \approx 8\times10^{-18}$.

Graph C - 1




Graph C - 2
15
10
5
- log Δb
15
10
5
- log d